

**Optimal Multiple Source Location  
via the EM Algorithm**

by

Meir Feder and Ehud Weinstein

Woods Hole Oceanographic Institution  
Woods Hole, Massachusetts 02543

July 1986

**Technical Report**

*Funding provided by the Naval Air Systems Command  
under contract Number N00014-85-K-0272.*

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Woods Hole Oceanog. Inst. Tech. Rept. WHOI-86-25.*

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**Robert C. Spindel, Chairman**  
Department of Ocean Engineering

# ABSTRACT

A computationally efficient scheme for multiple-source location estimation, based on the Estimate-Maximize (EM) algorithm [1], is presented. The proposed scheme is optimal in the sense that it converges iteratively to the exact Maximum Likelihood estimate of all source location parameters simultaneously. Versions of the algorithm that incorporate the estimation of the unknown amplitude attenuations and the estimation of the unknown signal waveforms are also presented.



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## I. INTRODUCTION

The location of a radiating point-source can be determined by observation of its signal at an array of spatially distributed sensors. The optimal, Maximum Likelihood (ML), estimate of the source location parameters (i.e. bearing and range) is achieved by maximizing the array beamformer output [2] or, alternatively, by cross-correlating the various sensor outputs and obtain the ML estimates by maximizing a weighted sum of the cross-correlation responses with respect to a pair of bearing and range parameters [3].

The presence of several signal sources drastically complicates the estimation process. To obtain the ML estimate of all source location parameters jointly, we have to maximize a significantly more complicated highly non-linear function with respect to  $K$  pairs of unknown location parameters, where  $K$  is the assumed number of signal sources. Of course, brute force can always be used to solve the problem, evaluating the objective function on coarse grid to roughly locate the global maximum, and then applying the Gauss method or Newton-Raphson or some other iterative gradient-search algorithm. However, when applied to the problem in hand, these methods tend to be very complex and computationally time consuming.

Consequently, approximations to the ML solution and various ad hoc schemes for multiple source localization have been proposed in recent literature (e.g., [4] - [20]); however, most of the proposed suboptimal localization schemes simplify processor structure and computations at the sacrifice of system resolution and accuracy.

In this report, we develop an iterative scheme for multiple source location estimation based on the Estimate-Maximize (EM) algorithm. However, unlike the brute force gradient-search iterative algorithms, the proposed scheme makes an essential use of the stochastic system under consideration.

The heuristic idea is to decompose the sum of vector signals observed at the sensor outputs into its components, and then apply a conventional beamformer instrumentation to each signal component to obtain the bearing and range estimate of the corresponding source. The algorithm iterates, using the resulting parameter estimates to better decompose the observed data on the next iteration cycle and thus to improve the next parameter estimates. This computationally attractive algorithm is shown to be optimal in the sense that it converges to the exact ML estimate of all source location parameters simultaneously.

The organization of the report is as follows: In Section II we re-derive the EM algorithm following the considerations in [1], and then we specialize to the Linear-Gaussian case. In Section III we apply the algorithm to the multiple source location estimation.

## II. MAXIMUM LIKELIHOOD ESTIMATION AND THE ESTIMATE-MAXIMIZE ALGORITHM

Let  $\underline{Y}$  be a vector random variables possessing the probability density  $f_Y(\underline{y}; \underline{\theta})$   $\underline{\theta} \in \Theta$ , where  $\Theta$  is an open subset of the K-dimensional Euclidean space. The ML estimate  $\hat{\underline{\theta}}_{ML}$  of  $\underline{\theta}$ , given an observed  $\underline{y}$ , is obtained by maximizing the log-likelihood function

$$\begin{aligned} \text{Max}_{\underline{\theta} \in \Theta} \log f_Y(\underline{y}; \underline{\theta}) &\Rightarrow \hat{\underline{\theta}}_{ML} \end{aligned} \quad (1)$$

If the vector  $\underline{\theta}$  contains several unknowns, such as in the multiple source location problem, and since  $\log f_Y(\underline{y}; \underline{\theta})$  is generally a highly non-linear function of  $\underline{\theta}$ , the maximization required in (1) tends to be very complex.

Suppose the data vector  $\underline{Y}$  can be viewed as being "incomplete", and we can specify some "complete" data  $\underline{X}$  that is related to  $\underline{Y}$  by

$$H(\underline{X}) = \underline{Y} \quad (2)$$

where  $H(\cdot)$  is a non-invertible (many-to-one) transformation. In the multiple source location problem, the "complete" data  $\underline{X}$  could be the observation of the various source signals separately, where the "incomplete" (observed) data  $\underline{Y}$  is the sum of the signal contributions from the various sources. As pointed out before, the  $\underline{Y}$  model may be complicated to work on directly, in which case reference to the  $\underline{X}$  model might be very useful.

For all "complete" data realizations  $\underline{X}$  such that  $H(\underline{X}) = \underline{y}$

$$f_{\underline{X}}(\underline{x}; \underline{\theta}) = f_{\underline{Y}}(\underline{y}; \underline{\theta}) \cdot f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta}) \quad (3)$$

where  $f_{\underline{X}}(\underline{x}; \underline{\theta})$  is the probability density of  $\underline{X}$ , and  $f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta})$  is the conditional probability density of  $\underline{X}$  given that  $\underline{Y}=\underline{y}$ . Taking the logarithm on both sides of (3), we obtain

$$\log f_{\underline{Y}}(\underline{y}; \underline{\theta}) = \log f_{\underline{X}}(\underline{x}; \underline{\theta}) - \log f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta}) \quad (4)$$

Taking the conditional expectation given  $\underline{Y}=\underline{y}$  for a parameter value  $\underline{\theta}'$  (i.e., multiplying both sides of (4) by  $f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta}')$  and integrating with respect to  $\underline{X}$  over  $\{\underline{x}/H(\underline{x})=\underline{y}\}$ ), we obtain

$$\begin{aligned} \log f_{\underline{Y}}(\underline{y}; \underline{\theta}) &= E \{ \log f_{\underline{X}}(\underline{x}; \underline{\theta}) / \underline{Y}=\underline{y}; \underline{\theta}' \} \\ &\quad - E \{ \log f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta}) / \underline{Y}=\underline{y}; \underline{\theta}' \} \end{aligned} \quad (5)$$

where we note that for a fixed  $\underline{y}$ , the left hand side of (5) is a constant and hence it is unaffected by the conditional expectation operation. Define for convenience

$$Q(\underline{\theta}, \underline{\theta}') = E \{ \log f_{\underline{X}}(\underline{x}; \underline{\theta}) / \underline{Y} = \underline{y}; \underline{\theta}' \} \quad (6)$$

$$P(\underline{\theta}, \underline{\theta}') = E \{ \log f_{\underline{X}/\underline{Y}=\underline{y}}(\underline{x}; \underline{\theta}) / \underline{Y} = \underline{y}; \underline{\theta}' \} \quad (7)$$

With these definitions, (5) reads

$$\log f_{\underline{Y}}(\underline{y}; \underline{\theta}) = Q(\underline{\theta}, \underline{\theta}') - P(\underline{\theta}, \underline{\theta}') \quad (8)$$

Using the Jensen's inequality, it can easily be verified that

$P(\underline{\theta}, \underline{\theta}') \leq P(\underline{\theta}', \underline{\theta}')$ \*). Hence

$$Q(\underline{\theta}, \underline{\theta}') > Q(\underline{\theta}', \underline{\theta}') \implies \log f_{\underline{Y}}(\underline{y}; \underline{\theta}) > \log f_{\underline{Y}}(\underline{y}; \underline{\theta}') \quad (9)$$

The relation in (9) form the basis to the EM algorithm. The algorithm starts with an initial guess  $\underline{\theta}^{(0)}$ , and let  $\underline{\theta}^{(n+1)}$  be defined inductively by

$$\underset{\underline{\theta}}{\text{Max}} Q(\underline{\theta}, \underline{\theta}^{(n)}) \implies \underline{\theta}^{(n+1)} \quad n=0,1,2,\dots \quad (10)$$

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\*) The Jensen's inequality asserts that for any two probability densities  $f(x)$  and  $g(x)$  defined over  $\Omega$

$$\int_{\Omega} f(x) \log g(x) dx \leq \int_{\Omega} g(x) \log g(x) dx$$

Since  $\underline{\theta}^{(n)}$  is the value of  $\underline{\theta}$  that maximizes  $Q(\underline{\theta}, \underline{\theta}^{(n)})$ , then according to (9) each iteration of the algorithm increases the likelihood. Hence, under the usual regularity conditions, the algorithm converges to the maximum likelihood estimate, i.e.  $\underline{\theta}^{(n)} \rightarrow \hat{\underline{\theta}}_{ML}$ . The rate of convergence of the algorithm is exponential (see [1]), depending on the fraction of the covariance of the "complete" data that can be predicted using the observed ("incomplete") data. If that fraction is small, the rate of convergence tends to be slow, in which case one could use standard numerical methods to accelerate the algorithm.

It is important to observe that the EM algorithm is not uniquely defined. The transformation  $H(\cdot)$  relating  $\underline{X}$  to  $\underline{Y}$  can be any non-invertible transformation. Obviously, there are many possible "complete" data specifications that will generate the observed data. Thus, the EM algorithm can be implemented in many possible ways. The final outcome, which is the ML estimate, is completely unaffected by the way in which  $H$  is specified (i.e. the choice of "complete" data); however, the choice of  $H$  may critically affect the complexity and rate of convergence of the algorithm, and unfortunate choice of  $H$  may yield a completely useless algorithm. Hence, given a parameter estimation model, the practically important question that is left open is how to find the computationally most efficient implementation of the algorithm.

#### The Linear-Gaussian Case

Suppose that  $\underline{H}\underline{X} = \underline{Y}$ , where  $\underline{H}$  is a  $m \times n$  matrix ( $m < n$ ), and  $\underline{X}$  possesses the following multivariate Gaussian probability density

$$f_{\underline{X}}(\underline{x}; \underline{\theta}) = \frac{1}{(\det[\frac{2\pi}{\lambda} R(\underline{\theta})])^{1/2}} \exp\left(-\frac{\lambda}{2} [\underline{x} - \underline{m}(\underline{\theta})]^* R^{-1}(\underline{\theta}) [\underline{x} - \underline{m}(\underline{\theta})]\right) \quad (11)$$

where  $\lambda=1$  if  $\underline{x}$  is real-valued,  $\lambda=2$  if  $\underline{x}$  is complex-valued, and the superscript  $*$  denotes the conjugate-transpose operator. We shall refer to this case as the Linear-Gaussian case. Taking the logarithm of (11), we obtain

$$\begin{aligned} \log f_X(\underline{x}; \underline{\theta}) &= -\frac{\lambda}{2} \log \det \left[ \frac{2\pi}{\lambda} R(\underline{\theta}) \right] - \frac{\lambda}{2} [\underline{x} - \underline{m}(\underline{\theta})]^* \tilde{R}(\underline{\theta}) [\underline{x} - \underline{m}(\underline{\theta})] \\ &= -\frac{\lambda}{2} \log \det \left[ \frac{2\pi}{\lambda} R(\underline{\theta}) \right] - \frac{\lambda}{2} \underline{m}^*(\underline{\theta}) \tilde{R}(\underline{\theta}) \underline{m}(\underline{\theta}) \\ &\quad + \frac{\lambda}{2} \underline{x}^* \tilde{R}(\underline{\theta}) \underline{m}(\underline{\theta}) + \frac{\lambda}{2} \underline{m}^*(\underline{\theta}) \tilde{R}(\underline{\theta}) \underline{x} - \frac{\lambda}{2} \text{tr} [\tilde{R}(\underline{\theta}) \underline{x} \underline{x}^*] \end{aligned} \quad (12)$$

where  $\text{tr} \underline{J}$  stands for the trace of the bracketed matrix. Substituting (12) into (6), we obtain

$$\begin{aligned} Q(\underline{\theta}, \underline{\theta}') &= -\frac{\lambda}{2} \log \det \left[ \frac{2\pi}{\lambda} R(\underline{\theta}) \right] - \frac{\lambda}{2} \underline{m}^*(\underline{\theta}) \tilde{R}(\underline{\theta}) \underline{m}(\underline{\theta}) \\ &\quad + \frac{\lambda}{2} \hat{\underline{x}}^* \tilde{R}(\underline{\theta}) \underline{m}(\underline{\theta}) + \frac{\lambda}{2} \underline{m}^*(\underline{\theta}) \tilde{R}(\underline{\theta}) \hat{\underline{x}} - \frac{\lambda}{2} \text{tr} [\tilde{R}(\underline{\theta}) \hat{\underline{x}} \hat{\underline{x}}^*] \end{aligned} \quad (13)$$

where  $\hat{\underline{x}} = E\{\underline{x}/H\underline{x}=\underline{y}; \underline{\theta}'\}$ , and  $\hat{\underline{x}}\hat{\underline{x}}^* = E\{\underline{x}\underline{x}^*/H\underline{x}=\underline{y}; \underline{\theta}'\}$ .

Using well-known results from linear estimation ([23], Chapter 4), we obtain

$$\hat{\underline{x}} = \underline{m}(\underline{\theta}') + \Gamma(\underline{\theta}') [\underline{y} - H \cdot \underline{m}(\underline{\theta}')] \quad (14)$$

$$\hat{\underline{x}}\hat{\underline{x}}^* = [I - \Gamma(\underline{\theta}') \cdot H] R(\underline{\theta}') + \hat{\underline{x}}\hat{\underline{x}}^* \quad (15)$$

where  $\Gamma(\underline{\theta})$  is the "Kalman Gain" defined by

$$\Gamma(\underline{\theta}) = R(\underline{\theta}) H^* [H R(\underline{\theta}) H^*]^{-1} \quad (16)$$



Substituting (14) and (15) into (13), the function  $Q(\underline{\theta}, \underline{\theta}')$  required for the EM algorithm is given in a closed form. We note that  $Q(\underline{\theta}, \underline{\theta}')$  and  $\log f_X(\underline{x}; \underline{\theta})$  have the same dependence on  $\underline{\theta}$ . Maximizing  $Q(\underline{\theta}, \underline{\theta}')$  with respect to  $\underline{\theta}$  is the same as maximizing  $\log f_X(\underline{x}; \underline{\theta})$  with respect to  $\underline{\theta}$ . Hence, the EM algorithm requires the ML solution in the  $\underline{X}$  model which might be significantly easier to obtain than the direct ML solution in the  $\underline{Y}$  model.

If  $R(\underline{\theta}) = R$  a constant matrix,  $Q(\underline{\theta}, \underline{\theta}')$  assumes the simplified form

$$Q(\underline{\theta}, \underline{\theta}') = a(\underline{\theta}') + \frac{\lambda}{2} [\hat{\underline{x}}^* R^{-1} \underline{m}(\underline{\theta}) + \underline{m}^*(\underline{\theta}) R^{-1} \hat{\underline{x}} - \underline{m}^*(\underline{\theta}) R^{-1} \underline{m}(\underline{\theta})] \quad (17)$$

where  $a(\underline{\theta}')$  accounts for all terms that are independent of  $\underline{\theta}$ . Substituting (17) into (10), the Estimate (conditional expectation) step and Maximize step of the algorithm are given by:

E-step: Compute

$$\underline{x}^{(n)} = \underline{m}(\underline{\theta}^{(n)}) + \Gamma(\underline{\theta}^{(n)}) [\underline{y} - H \cdot \underline{m}(\underline{\theta}^{(n)})] \quad (18)$$

M-step:

$$\underset{\underline{\theta}}{\text{Max}} [ \underline{x}^{(n)*} R^{-1} \underline{m}(\underline{\theta}) + \underline{m}^*(\underline{\theta}) R^{-1} \underline{x}^{(n)} - \underline{m}^*(\underline{\theta}) R^{-1} \underline{m}(\underline{\theta}) ] \Rightarrow \underline{\theta}^{(n+1)} \quad (19)$$

### III. ARRAY PROCESSING VIA THE EM ALGORITHM

The basic system of interest consists of K spatially distributed signal sources and an array of M spatially distributed sensors as illustrated in Figure 1. Assuming perfect propagation conditions in the medium between sources and receivers, the actual waveform observed at the  $m^{\text{th}}$  sensor output is given by

$$y_m(t) = \sum_{k=1}^K \alpha_{km} s_k(t - \tau_{km}) + n_m(t) \quad (20)$$

where  $s_k(t)$  is the  $k^{\text{th}}$  source signal,  $\mathcal{N}_m(t)$  is the additive noise,  $\tau_{km}$  is the travel time of the signal wavefront from the  $k^{\text{th}}$  source to the  $m^{\text{th}}$  sensor, and  $\alpha'_{km}$  are the amplitude attenuations.

Information concerning the various source location parameters can be extracted by measuring the various  $\tau_{km}$ . In the passive case, one can only measure the travel time differences, obtainable by selecting one sensor as a reference and comparing its output with that of every other sensor. If we let sensor  $M$  to be the reference and set  $\tau_{kM} = 0$ , then  $\tau_{km}$  measures the travel time difference of the  $k^{\text{th}}$  signal wavefront to the  $(m, M)$  sensor pair.

To simplify the exposition, suppose that the various signal sources are relatively far-field so that the observed signal wavefronts are essentially planar across the array. If we further suppose that the array sensors are co-linear, then

$$\tau_{km} = \frac{d_m}{c} \cos \psi_k \quad (21)$$

where  $d_m$  is the spacing between sensors  $m$  and  $M$ ,  $c$  is the velocity of propagation in the medium, and  $\psi_k$  is the angle of incident of the  $k^{\text{th}}$  signal wavefront with respect to the array baseline.

In this setting, the estimation problem can be stated as follows: Given the observed data  $\{y_m(t)\}_{m=1}^M$ , find the ML estimate of  $\psi_1, \psi_2, \dots, \psi_K$ . We shall find it convenient to work with the parameters  $\theta_k = \cos \psi_k$ . Since ML estimation commutes over non-linear transformation, we can first estimate the  $\theta_k$  and then translate to the  $\psi_k$ .

Assuming that the  $s_k(t)$  are perfectly known to the observer, and that the  $\mathcal{N}_m(t)$  are realizations from uncorrelated zero-mean spectrally white Gaussian processes, the log-likelihood function is given by

$$\log f_Y(y; \theta) = G - \sum_{m=1}^M \frac{1}{N_m} \int_{T_1}^{T_2} \left[ y_m(t) - \sum_{k=1}^K \alpha'_{km} s_k(t - \tau_{km} \theta_k) \right]^2 dt \quad (22)$$

where  $\gamma_m = d_m/c$ ,  $N_m$  is the spectral level of  $\eta_m(t)$ ,  $[T_i, T_f]$  is the observation interval, and  $C$  is a normalizing constant. The result in (22) is a straightforward multi-channel extension of the known (deterministic) signal in white Gaussian noise problem ([24], Chapter 4). Thus, the ML estimate is the solution to the following problem:

$$\text{Max}_{\theta_1, \theta_2, \dots, \theta_K} \left\{ - \sum_{m=1}^M \frac{1}{N_m} \int_{T_i}^{T_f} \left[ y_m(t) - \sum_{k=1}^K \alpha_{km} s_k(t - \gamma_m \theta_k) \right]^2 dt \right\} \rightarrow \hat{\underline{\theta}}_{ML} \quad (23)$$

Ignoring terms that are independent of  $\underline{\theta}$ , (23) reduces to

$$\begin{aligned} \text{Max}_{\theta_1, \theta_2, \dots, \theta_K} \left\{ \sum_{m=1}^M \frac{1}{N_m} \left[ 2 \cdot \sum_{k=1}^K \alpha_{km} \int_{T_i}^{T_f} y_m(t) s_k(t - \gamma_m \theta_k) dt \right. \right. \\ \left. \left. - \sum_{k=1}^K \sum_{l=1}^K \alpha_{km} \alpha_{lm} \int_{T_i}^{T_f} s_k(t - \gamma_m \theta_k) s_l(t - \gamma_m \theta_l) dt \right] \right\} \Rightarrow \hat{\underline{\theta}}_{ML} \quad (24) \end{aligned}$$

In many situations of practical interest, the number of signal sources is also an unknown parameter to be estimated, in which case several criteria to determine  $K$ , based on the ML estimate of the  $\theta_k$ 's, are presented in ([20] - [22]). Thus, to obtain the ML estimate of the  $\theta_k$ 's jointly with the estimate of the number of signal sources, we have to solve a non-linear

optimization problem in K unknowns, for which a closed form analytical solution cannot be found.

We further note that the optimization problem stated above assumes prior knowledge of the  $\alpha_{km}'s$  and  $S_k(t)'s$ . In practice, this is apt to be unrealistic. One is unlikely to have exact prior knowledge of the amplitude attenuations and detailed description of the signal waveshapes may be similarly incomplete, in which case to obtain the ML estimate of the  $\theta_k's$  we must maximize the expression in (24) with respect to all the unknowns in the problem. The effect of unknown attenuation factors can be eliminated from the likelihood equation by observing that for pre-specified  $\theta_k's$ , (23) becomes a weighted linear least squares problem in the  $\alpha_{km}'s$ , for which there is a closed form solution. Thus, we can substitute the  $\alpha_{km}'s$  by their weighted least squares estimates and obtain a somewhat more complicated functional to be maximized with respect to the remaining unknowns. However, the effect of unknown signal waveshapes cannot be eliminated that easily and hence, the required maximization becomes exceedingly more complicated.

Having the EM method in mind, we would like to simplify the maximization associated with the direct ML approach. To apply the algorithm to the problem in hand, the first step is to specify the "complete" data. A natural choice of the "complete" data is given by decomposing the observed signals  $y_m(t)$  into

$$X_{bm}(t) = \alpha_{bm} S_b(t - \tau_{bm}) + \eta_{bm}(t) \quad (25)$$

where the  $\eta_{bm}(t)$  are chosen to be realizations from uncorrelated zero-mean white Gaussian processes with spectral levels of  $N_{bm} = \beta_{bm} N_m$ . If we require that

$$\sum_{b=1}^K \eta_{bm}(t) = \eta_m(t) \quad (26)$$

then

$$\sum_{k=1}^K X_{km}(t) = y_m(t) \quad (27)$$

Concatenating the  $M$  equations in a vector form, we obtain

$$\sum_{k=1}^K \underline{X}_k(t) = \underline{y}(t) \quad (28)$$

where

$$\underline{X}_k(t) = (X_{k1}(t) \ X_{k2}(t) \ \dots \ X_{kM}(t))^T \quad (29)$$

and

$$\underline{y}(t) = (y_1(t) \ y_2(t) \ \dots \ y_M(t))^T \quad (30)$$

Equation (28) can be rewritten in the form

$$H \cdot \underline{X}(t) = \underline{y}(t) \quad (31)$$

where

$$\underline{X}(t) = ( \underline{X}_1^T(t) \underline{X}_2^T(t) \dots \underline{X}_K^T(t) ) \quad (32)$$

and

$$H = \underbrace{[I \mid I \mid \dots \mid I]}_{K \text{ terms}} \quad (33)$$

where  $I$  is the  $M \times M$  identity matrix. We denote  $\underline{X}(t)$  the "complete" data, and  $\underline{Y}(t)$  the "incomplete" (observed) data. Note that  $\underline{X}(t)$  is composed of the  $K$  components  $\underline{X}_k(t)$   $k=1, 2, \dots, K$ , where  $\underline{X}_k(t)$  is the vector signals received from the  $k^{\text{th}}$  source alone.

Consider, for the moment, ML estimation using the "complete" data. Since the components of  $\underline{X}(t)$  are uncorrelated Gaussian random processes, the log-likelihood of  $\underline{X}(t)$  is the sum over  $k$  and  $m$  of the log-likelihood of the  $X_{km}(t)$  which, in turn, is given by

$$\log / \underline{X}(\underline{x}; \underline{\theta}) = C - \sum_{k=1}^K \sum_{m=1}^M \frac{1}{N_{km}} \int_{T_1}^{T_2} [X_{km}(t) - \alpha_{km} S_k(t) - \tilde{r}_m \theta_k]^2 dt \quad (34)$$

Suppose that the  $S_k(t)$  are known a-priori so that we "only" have to estimate the  $K$  sets  $\{ \theta_k, \alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kM} \}_{k=1}^K$ . The maximization of (24) with respect to all unknowns is a rather complex maximization problem. The maximization of (34), however, is a much simpler problem. This is since the expression in (34) can be decomposed into

$$\log / \underline{X}(\underline{x}; \underline{\theta}) = \sum_{k=1}^K \log / \underline{X}_k(\underline{x}_k; \underline{\theta}_k) \quad (35)$$

where

$$\log f_{\underline{X}_k}(\underline{y}_k; \underline{\theta}_k) = C_k - \sum_{m=1}^M \frac{1}{N_{km}} \int_{T_i}^{T_f} [y_{km}(t) - \alpha_{km} s_k(t - r_m \theta_k)]^2 dt \quad (36)$$

The expression in (36) depends only on  $\theta_k, \alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kM}$ . Hence, the maximization of (34) with respect to the K-set unknowns can be decomposed into the K separate maximizations

$$\begin{aligned} \text{Max}_{\theta_k, \alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kM}} \left\{ - \sum_{m=1}^M \frac{1}{N_{km}} \int_{T_i}^{T_f} [y_{km}(t) - \alpha_{km} s_k(t - r_m \theta_k)]^2 dt \right\} = \\ = \text{Max}_{\theta_k, \alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kM}} \sum_{m=1}^M \frac{1}{N_{km}} \left[ 2\alpha_{km} \int_{T_i}^{T_f} y_{km}(t) s_k(t - r_m \theta_k) dt - \alpha_{km}^2 \int_{T_i}^{T_f} s_k^2(t - r_m \theta_k) dt \right] \end{aligned} \quad (37)$$

The maximization in (37) is, in fact, the maximization problem associated with the ML estimation of the bearing parameter of a single source in the presence of unknown amplitude attenuations.

The EM algorithm is directed at finding a value of the parameters that maximize (24), however, it does so by making an essential use of the solution to (37).

Since the "complete" data is Gaussian, and the transformation relating  $\underline{X}(t)$  to  $\underline{Y}(t)$  is linear, we can use the version of the algorithm developed for the Linear-Gaussian case. Thus, the M-step of the algorithm is given by (34) (since  $Q(\underline{y}, \underline{\theta}')$  and  $\log f_{\underline{X}}(\underline{y}; \underline{\theta})$  assumes the same form), where the components of  $\underline{X}(t)$  are substituted by

$$\underline{X}^{(n)}(t) = E \{ \underline{X}(t) / H \underline{X}(t) = \underline{Y}(t); \underline{\theta}^{(n)} \} \quad (38)$$

The conditional expectation required in (38) can easily be obtained using (1b). The resulting algorithm is given by:

E-step

Compute

$$X_{km}^{(n)}(t) = \alpha_{km}^{(n)} s_k(t - r_m \theta_k^{(n)}) + \beta_{km} \left[ y_m(t) - \sum_{\ell=1}^K \alpha_{\ell m}^{(n)} s_{\ell}(t - r_m \theta_{\ell}^{(n)}) \right] \quad (39)$$

M-step

For

$$k = 1, 2, \dots, K$$

$$\begin{aligned} \text{Max}_{\theta_k, \alpha_{k1}, \dots, \alpha_{kM}} \sum_{m=1}^M \frac{1}{N_{km}} \left[ 2\alpha_{km} \int_{T_1}^{T_2} y_{km}^{(n)}(t) s_k(t - r_m \theta_k) dt \right. \\ \left. - \alpha_{km}^2 \int_{T_1}^{T_2} s_k^2(t - r_m \theta_k) dt \right] \Rightarrow \theta_k^{(n+1)}, \alpha_{k1}^{(n+1)}, \dots, \alpha_{kM}^{(n+1)} \end{aligned} \quad (40)$$

Perhaps the most striking feature of the algorithm has already been indicated before. The algorithm decouples the complex multiple-parameter maximization associated with the direct ML procedure into K separate ML maximizations as illustrated in Figure 2. The extension to bearing and range estimation is straightforward. The basic scheme is still illustrated by Figure 2, where now each ML processor requires the maximization with respect to a pair of bearing and range parameters. Thus, the complexity of the algorithm is completely unaffected by the number of signal sources. As the number of sources increases, we have to increase the number of ML processors in parallel; however, each processor is maximized separately.



Since the algorithm is based on the EM method, it must converge to the exact ML estimate of all source location parameters simultaneously.

We note, in passing, that according to (26), the  $\beta_{km}$  must satisfy the constraint

$$\sum_{k=1}^K \beta_{km} = 1 \quad (41)$$

but otherwise they are free variables. The choice of  $\beta_{km}$  does not affect the final estimate (at the point of convergence); however, it can be used to control the rate of convergence of the algorithm.

We now consider the M-step of the algorithm in more details. If we set the derivative of the expression in (40) equals to zero, we obtain

$$\alpha'_{km} = \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - r_m \theta_k) dt}{\int_{T_i}^{T_f} S_k^2(t - r_m \theta_k) dt} \quad (42)$$

Since the second derivatives with respect to the  $\alpha'_{km}$  is a negative definite matrix, we have therefore expressed the optimal choice of the  $\alpha'_{km}$  as a function of  $\theta_k$ . Substituting (42) into (40) and following straightforward algebra manipulations, the M-step of the algorithm reduces to

$$\text{Max}_{\theta_k} \sum_{m=1}^M \left( \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - r_m \theta_k^{(n+1)}) dt}{N_{km} \int_{T_i}^{T_f} S_k^2(t - r_m \theta_k) dt} \right)^2 \Rightarrow \theta_k^{(n+1)} \quad (43)$$

$$\alpha_{km}^{(n+1)} = \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - r_m \theta_k^{(n+1)}) dt}{\int_{T_i}^{T_f} S_k^2(t - r_m \theta_k^{(n+1)}) dt} \quad (44)$$

In most situations of practical interest, we can assume that

$\int_{T_i}^{T_f} S_k^2(t - \tau_m \theta_k) dt$  is independent of  $\theta_k$ , in which case the above pair of equations reduces to

$$\text{Max}_{\theta_k} \sum_{n=1}^M \frac{1}{N_{km}} \left[ \int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - \tau_m \theta_k) dt \right]^2 \Rightarrow \theta_k^{(n+1)} \quad (45)$$

$$\alpha_{km}^{(n+1)} = \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - \tau_m \theta_k^{(n+1)}) dt}{\int_{T_i}^{T_f} S_k^2(t) dt} \quad (46)$$

The term  $\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - \tau) dt$  can be generated by passing  $X_{km}^{(n)}(t)$  through a filter matched to  $S_k(t)$ . Thus,  $\theta_k^{(n+1)}$  is obtained by maximizing a squared and weighted sum of a bank match filter outputs.

#### Modified EM Algorithm

The EM theory allows us to substitute the M-step of the algorithm (Equation (40)) by the following two-step maximization:

$$\text{Max}_{\theta_k} \sum_{n=1}^M \frac{1}{N_{km}} \left[ 2\alpha_{km}^{(n)} \int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - \tau_m \theta_k) dt - \alpha_{km}^{(n)2} \int_{T_i}^{T_f} S_k^2(t - \tau_m \theta_k) dt \right] \Rightarrow \theta_k^{(n+1)} \quad (47)$$

$$\text{Max}_{\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kM}} \sum_{n=1}^M \frac{1}{N_{km}} \left[ 2\alpha_{km} \int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - \tau_m \theta_k^{(n+1)}) dt - \alpha_{km}^2 \int_{T_i}^{T_f} S_k^2(t - \tau_m \theta_k^{(n+1)}) dt \right] \Rightarrow \alpha_{k1}^{(n+1)}, \dots, \alpha_{kM}^{(n+1)} \quad (48)$$

Assuming, as before, that  $\int_{T_i}^{T_f} S_k^2(t - V_m \theta_k) dt$  is independent of  $\theta_k$ , (47) and (48) reduce to

$$\text{Max}_{\theta_k} \sum_{m=1}^M \frac{\alpha_{km}^{(n)}}{N_{km}} \int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - V_m \theta_k) dt \Rightarrow \theta_k^{(n+1)} \quad (49)$$

$$\alpha_{km}^{(n+1)} = \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - V_m \theta_k^{(n+1)}) dt}{\int_{T_i}^{T_f} S_k^2(t) dt} \quad (50)$$

We note that the solution to (49) is obtained by maximizing a weighted linear sum of a bank match filter outputs.

The maximization of (47) followed by the maximization of (48) is generally not equivalent to the maximization required in (40). However, since the condition in (9) is still satisfied, each iteration increases the likelihood, and the modified algorithm converges to the desired results. By replacing (43) by (49), we therefore simplify processor structure and computations in the expense of possibly very moderate decrease in the rate of convergence of the algorithm.

Another alternative is to maximize the expression in (40) first with respect to the  $\alpha_{km}'$  (using  $\theta_k^{(n)}$ ) and then with respect to  $\theta_k$ . The M-step of the algorithm, in that case, is given by

$$\alpha_{km}^{(n+1)} = \frac{\int_{T_i}^{T_f} X_{km}^{(n)}(t) S_k(t - V_m \theta_k^{(n)}) dt}{\int_{T_i}^{T_f} S_k^2(t) dt} \quad (51)$$

$$\text{Max}_{\theta_k} \sum_{m=1}^M \frac{\alpha_{km}^{(n+1)}}{N_{km}} \int_{T_i}^{T_f} x_{km}^{(n)}(t) S_k(t - \tau_m \theta_k) dt \Rightarrow \theta_k^{(n+1)} \quad (52)$$

We note that (49) followed by (50) is not the same as (51) followed by (52). However, for the same reason, both algorithms converge to the desired ML solution.

### Unknown Signal waveforms

The algorithm and its modified version developed in the previous sections critically depends on exact prior knowledge of the  $S_k(t)$ . In practice, one is unlikely to have detailed prior knowledge of the signal waveforms, in which case they must be estimated jointly with the  $\theta_k$  and the  $\alpha'_{km}$ . Following the same considerations as in the known signal case, the resulting algorithm is given by

### E-step

Compute

$$x_{km}^{(n)}(t) = \alpha_{km}^{(n)} S_k^{(n)}(t - \tau_m \theta_k^{(n)}) + \beta_{km} \left[ y_m(t) - \sum_{l=1}^K \alpha_{lm}^{(n)} S_l^{(n)}(t - \tau_m \theta_l^{(n)}) \right] \quad (53)$$

### M-step

For  $k = 1, 2, \dots, K$

$$\begin{aligned} & \text{Max}_{\theta_k, S_k(t), \alpha_{k1}, \dots, \alpha_{kM}} \sum_{m=1}^M \frac{1}{N_{km}} \left[ 2\alpha_{km} \int_{T_i}^{T_f} x_{km}^{(n)}(t) S_k(t - \tau_m \theta_k) dt \right. \\ & \left. - \alpha_{km}^2 \int_{T_i}^{T_f} S_k^2(t - \tau_m \theta_k) dt \right] \Rightarrow \theta_k^{(n+1)}, S_k^{(n+1)}(t), \alpha_{k1}^{(n+1)}, \dots, \alpha_{kM}^{(n+1)} \quad (54) \end{aligned}$$

Again, the EM algorithm converts the extremely complicated maximization problem associated with the direct ML procedure into K separate maximizations as suggested by Figure 2. Each maximization is, in fact, the maximization problem associated with the ML estimation of the bearing parameter of a single source of unknown waveshape in the presence of unknown amplitude attenuations.

Ignoring end-effects, the expression in (54) to be maximized can be written in the form

$$\int_{T_1}^{T_2} \left( \sum_{m=1}^M \frac{a_{km}}{N_{km}} \left[ 2 Y_{km}^{(m)}(t + r_m \theta_k) S_k(t) - a_{km} S_k^2(t) \right] \right) dt \quad (55)$$

To find a function  $S_k(t)$  that maximizes the integral given in (55), it is sufficient to find  $S_k(t)$  that maximizes the integrand. If we set the derivative of the integrand with respect to  $S_k(t)$  equals to zero, we obtain after some obvious manipulations

$$S_k(t) = \frac{\sum_{m=1}^M a_{km} Y_{km}^{(m)}(t + r_m \theta_k) / N_{km}}{\sum_{m=1}^M a_{km}^2 / N_{km}} \quad (56)$$

Since the second derivative of the integrand with respect to  $S_k(t)$  is negative, we have found the optimal choice of  $S_k(t)$ , expressed in terms of the remaining unknowns in the problem. Substituting (56) into (54), the required maximization can be carried out as following:

$$\text{Max}_{\theta_k, d_{k1}, \dots, d_{kM}} \left\{ \sum_{m=1}^M \sum_{l=1}^M \frac{d_{km} d_{kl}}{N_{km} N_{kl}} \int_{T_i}^{T_f} x_{km}^{(n)}(t + \tau_m \theta_k) y_{kl}^{(n)}(t + \tau_l \theta_k) dt / \sum_{m=1}^M d_{km}^2 / N_{km} \right\} \Rightarrow \theta_k^{(n+1)}, d_{k1}^{(n+1)}, \dots, d_{kM}^{(n+1)} \quad (57)$$

$$S_k^{(n+1)}(t) = \frac{\sum_{m=1}^M d_{km}^{(n+1)} x_{km}^{(n)}(t + \tau_m \theta_k^{(n+1)}) / N_{km}}{\sum_{m=1}^M (d_{km}^{(n+1)})^2 / N_{km}} \quad (58)$$

We note that at convergence, equation (58) yields the ML estimate of the  $S_k(t)$  as well. This information is of considerable interest in many applications.

The maximization required in (57) is still rather complicated. However, the modified EM approach can be used to further simplify the computations by maximizing (54) first with respect to  $\theta_k$ , then with respect to  $S_k(t)$ , and finally with respect to the  $d_{km}$ . The M-step of the algorithm, in that case, takes the form:

$$\text{Max}_{\theta_k} \sum_{m=1}^M \sum_{l=1}^M \frac{d_{km}^{(n)} d_{kl}^{(n)}}{N_{km} N_{kl}} \int_{T_i}^{T_f} x_{km}^{(n)}(t + \tau_m \theta_k) x_{kl}^{(n)}(t + \tau_l \theta_k) dt \Rightarrow \theta_k^{(n+1)} \quad (59)$$

$$S_k^{(n+1)}(t) = \frac{\sum_{m=1}^M d_{km}^{(n)} x_{km}^{(n)}(t + \tau_m \theta_k^{(n+1)}) / N_{km}}{\sum_{m=1}^M (d_{km}^{(n)})^2 / N_{km}} \quad (60)$$

$$d_{km}^{(n+1)} = \frac{\int_{T_i}^{T_f} x_{km}^{(n)}(t) S_k^{(n+1)}(t - \tau_m \theta_k^{(n+1)}) dt}{\int_{T_i}^{T_f} [S_k^{(n+1)}(t)]^2 dt} \quad (61)$$

In this setting we only have to perform a one-parameter maximization at each iteration cycle. The expression in (59), to be maximized, can be rewritten in the form:

$$\begin{aligned} & \sum_{m=1}^M \sum_{l=1}^L \frac{d_{km}^{(n)} d_{le}^{(n)}}{N_{km} N_{le}} \int_{T_1}^{T_2} y_{km}^{(n)}(t + r_m \theta_k) y_{le}^{(n)}(t + r_l \theta_k) dt \\ &= \int_{T_1}^{T_2} \left[ \sum_{m=1}^M \frac{d_{km}^{(n)}}{N_{km}} x_{km}^{(n)}(t + r_m \theta_k) \right]^2 dt \\ &= \frac{1}{2\pi} \int_W \left| \sum_{m=1}^M \frac{d_{km}^{(n)}}{N_{km}} X_{km}(\omega) e^{j\omega r_m \theta_k} \right|^2 d\omega \end{aligned} \quad (62)$$

where  $X_{km}(\omega)$  is the Fourier transform of  $x_{km}(t)$  and  $W$  is the signal frequency band. The expression in (62) is the array beamformer, implemented in either the time or the frequency domain. Thus, the  $M$ -step of the algorithm essentially consists of maximizing  $K$  beamformers in parallel as illustrated in Figure 5.

Alternatives to the  $M$ -step can be obtained by changing the order of maximizations. For example, we can apply the various algorithms derived for the known signal case, where  $s_k(t)$  is substituted by its current estimate  $\hat{s}_k^{(n)}(t)$ , and then use (60) to update the estimate of  $s_k(t)$ . The various algorithms may have slightly different convergence properties; however, all of which converge to the exact ML estimate of all source location parameters simultaneously.

### COMMENTS

1) The extension of the algorithms to bearing and range estimation and arbitrary (but known) array configurations is straightforward. 2) The derivation of the algorithms for the case in which the  $S_k(t)$  are modelled as sample functions from uncorrelated zero-mean Gaussian random processes is presented in [25]. 3) In the assumed model, we have ignored the phase shifts caused by scattering and reflection phenomena. These effects can be taken into account by considering the  $y_m(t)$  in (20) to be the complex envelope of the received signals, in which case the  $\alpha_{km}$  are complex-valued (magnitude and phase) amplitude attenuations. The extension of the algorithms to the complex case is straightforward (note that the Linear-Gaussian case has been developed for complex processes as well). 4) The proposed scheme can be extended to time delay and location estimation in multipath environment.



ACKNOWLEDGEMENTS

This study has been supported by The Naval Air Systems Command under contract N00014-85-K-0272. The authors wish to thank Ms. Cindy Leonard for her excellent secretarial assistance.

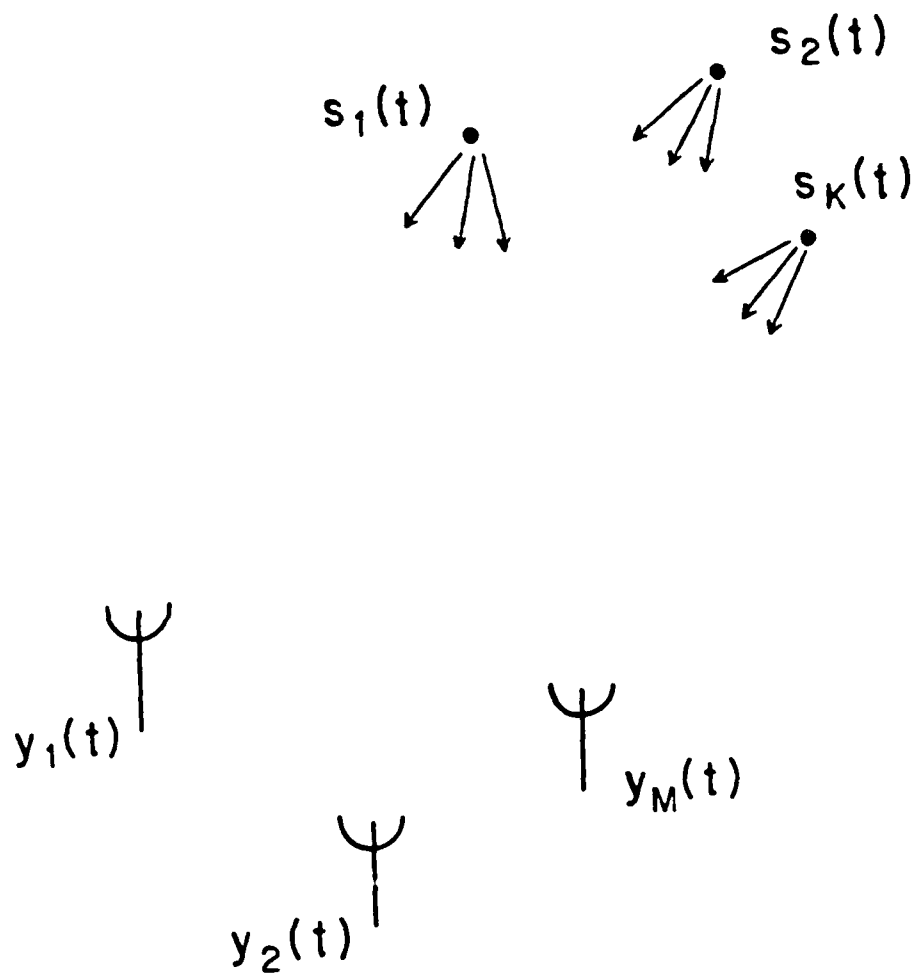
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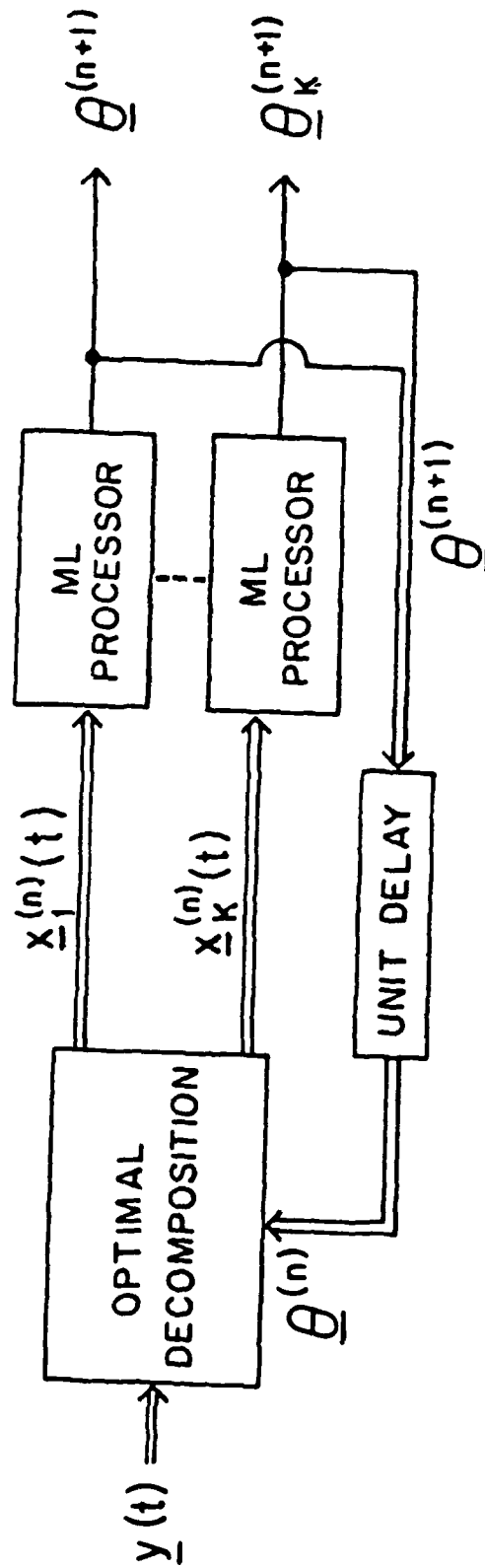
FIGURE CAPTIONS

- Figure 1            Array-Source Geometry
- Figure 2            Multiple Source Localization via the EM Algorithm
- Figure 3            Localization of Multiple Sources Having Unknown  
Waveforms



ARRAY - SOURCE GEOMETRY

Figure 1



MULTIPLE SOURCE LOCALIZATION  
VIA THE EM ALGORITHM

Figure 2

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<b>REPORT DOCUMENTATION PAGE</b>		<b>1. REPORT NO.</b> WHOI-86-25	<b>2.</b>	<b>3. Recipient's Accession No.</b>
<b>4. Title and Subtitle</b> Optimal Multiple Source Location via the EM Algorithm				<b>5. Report Date</b> July 1986
<b>7. Author(s)</b> Meir Feder and Ehud Weinstein				<b>6.</b>
<b>9. Performing Organization Name and Address</b>  Woods Hole Oceanographic Institution Woods Hole, Massachusetts 02543				<b>8. Performing Organization Rept. No.</b> WHOI-86-25
<b>12. Sponsoring Organization Name and Address</b>  Naval Air Systems Command				<b>10. Project/Task/Work Unit No.</b>
				<b>11. Contract(C) or Grant(G) No.</b> (C) N00014-85-K-0272 (G)
				<b>13. Type of Report &amp; Period Covered</b>  Technical
				<b>14.</b>
<b>15. Supplementary Notes</b> This report should be cited as: Woods Hole Oceanog. Inst. Tech. Rept. WHOI-86-25.				
<b>16. Abstract (Limit: 200 words)</b>  A computationally efficient scheme for multiple-source location estimation, based on the Estimate-Maximize (EM) algorithm [1], is presented. The proposed scheme is optimal in the sense that it converges iteratively to the <u>exact Maximum Likelihood</u> estimate of all source location parameters simultaneously. Versions of the algorithm that incorporate the estimation of the unknown amplitude attenuations and the estimation of the unknown signal waveforms are also presented.				
<b>17. Document Analysis a. Descriptors</b> 1. Array Processing 2. Multiple Source 3. EM Algorithm  <b>b. Identifiers/Open-Ended Terms</b>   <b>c. COSATI Field/Group</b>				
<b>18. Availability Statement:</b>  Approved for publication; distribution unlimited.		<b>19. Security Class (This Report)</b> UNCLASSIFIED		<b>21. No. of Pages</b> 29
		<b>20. Security Class (This Page)</b>		<b>22. Price</b>